

Massive graviton propagation of the deformed Hořava-Lifshitz gravity without projectability condition

Yun Soo Myung^a

*Institute of Basic Science and School of Computer Aided Science
Inje University, Gimhae 621-749, Korea*

Abstract

We study graviton propagations of scalar, vector, and tensor modes in the deformed Hořava-Lifshitz gravity (λR -model) without projectability condition. The quadratic Lagrangian is invariant under diffeomorphism only for $\lambda = 1$ case, which contradicts to the fact that λ is irrelevant to a consistent Hamiltonian approach to the λR model. In this case, as far as scalar propagations are concerned, there is no essential difference between deformed Hořava-Lifshitz gravity (λR -model) and general relativity. This implies that there are two degrees of freedom for a massless graviton without Hořava scalar, and five degrees of freedom appear for a massive graviton when introducing Lorentz-violating and Fierz-Pauli mass terms. Finally, it is shown that for $\lambda = 1$, the vDVZ discontinuity is absent in the massless limit of Lorentz-violating mass terms by considering external source terms.

^aysmyung@inje.ac.kr

1 Introduction

Recently Hořava has proposed a renormalizable theory of gravity at a Lifshitz point [1], which may be regarded as a UV complete candidate for general relativity. Very recently, the Hořava-Lifshitz gravity with a flow parameter λ has been intensively investigated in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. There are two versions of Hořava-Lifshitz gravity in the literature: with/without the projectability condition [13]. Hořava has originally proposed the projectability condition with/without the detailed balance condition. We mention that the IR vacuum of this theory is anti de Sitter (AdS) spacetimes. Hence, it is interesting to take a limit of the theory, which leads to a Minkowski vacuum in the IR limit. To this end, one may modify the theory by including “ $\mu^4 R$ ” and then, taking the $\Lambda_W \rightarrow 0$ limit [6]. This deformed Hořava-Lifshitz (dHL) gravity does not alter the UV properties of the theory. We note that the dHL gravity is composed of λR -model and higher spatial derivative terms from detailed balance condition. As far as the scalar propagations are concerned, the essential part is the λR -model because most issues arose from this model.

Concerning the projectability condition, its role should be dealt with carefully. Actually, there exists a close relation between projectability condition and scalar degrees of freedom. The projectability condition requires that the perturbation A of the lapse function N depends only on time. It means that $A = A(t)$ is not a Lagrange multiplier but a parameter. More seriously, by imposing this condition at the beginning, one found the global Hamiltonian constraint instead of the local one. This implies that with the projectability condition, the general relativity could not be recovered from the dHL gravity with any λ .

An urgent issue of the dHL gravity is still to answer to the question of whether it can accommodate the Hořava scalar ψ , in addition to two physical degrees of freedom (DOF) for a massless graviton. We would like to mention a few of relevant works. The authors [7] have shown that without the projectability condition, the Hořava scalar ψ is related to a scalar degree of freedom appeared in the massless limit of a massive graviton. Especially for the Hamiltonian approach to the dHL gravity, the authors [14] did not consider the Hamiltonian constraint as a second class constraint, which leads to a strange result that there are no DOF left when imposing the constraints of the theory. Moreover, the authors [15] have claimed that there are no solution of the lapse function which satisfies the constraints. Unfortunately, it implies a surprising conclusion that there is no evolution at all for any observable. More recently, it was shown that the λR -model (IR version of dHL gravity) which is considered as a gauge-fixed version of general relativity is equivalent to the general relativity for any λ when employing a consistent Hamiltonian formalism based on the Dirac algorithm [16, 17]. Although these has made a progress toward a consistent Hamiltonian

approach to the dHL gravity, there remains a subtle issue on the equivalence¹

With the projectability condition, the authors [8, 10] have argued that ψ is propagating around the Minkowski space but it has a negative kinetic term, showing a ghost instability. In this case, the Hořava scalar becomes ghost if the sound speed square (c_ψ^2) is positive. In order to avoid a ghost instability, the sound speed square must be negative, but it is inevitably unstable (gradient instability). Thus, one way to avoid this is to choose the case that the sound speed square is close to zero ($c_\psi^2 \rightarrow 0$), which implies the limit of $\lambda \rightarrow 1$. Unfortunately, in the limit of $\lambda \rightarrow 1$, the cubic interactions are important at very low energies [18]. This invalidates any linearized analysis and any predictability of quantum gravity is lost due to unsuppressed loop corrections. This strong coupling problem appears for an interacting theory of dHL gravity beyond the linearized theory. This casts serious doubts on the UV completeness of the theory. Also, it was shown that adding the mass term does not cure a ghost instability in the Hořava scalar [19]. However, it was suggested that there are many ways to tame the gradient instability of Hořava scalar [20]. These are included (i) the time scale is required to be longer than either the Jeans time scale or the Hubble time scale (ii) higher spatial derivatives would stabilize this instability when considering the dispersion relation (iii) a phenomenological constraint on the renormalization group flow may resolve the instability.

On the other hand, the authors [21] have tried to extend the theory to make a healthy Hořava-Lifshitz gravity. However, there has been some debate as to whether this theory is really healthy. The authors [22] considered the IR limit of this theory and showed that it suffered from the strong coupling problem, too. To response it, the original authors [23] have claimed that the strong coupling scale might exceed the cut-off scale for the derivative expansions and thus, it seems to be no strong coupling issue. More recently, the authors [24] has argued that the alleged strong coupling problem is genuine and not merely an artifact of a truncation the derivative expansion.

Hence, a current status of the dHL gravity may be summarized: the projectability condition from condensed matter physics may not be appropriate for describing the (quantum) gravity. Instead, if one does not impose the projectability condition, the dHL gravity may lead to general relativity without the strong coupling problem in the IR limit.

¹For example, one may find a vacuum torus universe of $N = 0$ [25] by assuming technical steps: First, Eq.(20) in Ref.[17] is multiplied by the lapse function N . Then, integrating Eq.(20) over a whole space and finally, requiring $R > 0$. This is confirmed from the vacuum Hamiltonian constraint Eq.(8) together with a second class constraint $\pi = 0$: $R > 0$ or $\pi_{ij} = 0$. This implies that there is no gravitational waves in the torus universe, which seems contrary to the general relativity. However, we have to admit that the torus universe is not an outcome of the consideration, but it appears as a result of assuming technical steps to avoid subtlety due to the boundary contribution. We thank anonymous referee for pointing out this point.

Inspired by a recent work of the consistency of the λR -model (IR version of dHL gravity) [17], we will perform a perturbation analysis of the dHL gravity without the projectability condition thoroughly. In this work, without the projectability condition, we investigate massive graviton propagations of scalar, vector, and tensor modes in the perturbation of dHL gravity by introducing Lorentz-violating mass term (18) and Fierz-Pauli mass term (19). A motivation of the introduction of these mass terms is to investigate the strong coupling problem and the vDVZ discontinuity. Even these mass terms violate the full diffeomorphism symmetry without the projectability condition, it provides more DOF through the spontaneous symmetry breaking: less symmetry means more degrees of freedom. Hence, we expect the change that 2 DOF (for massless theory) \rightarrow 5 DOF (for massive theory) including the Hořava scalar. We will show that the strong coupling problem is not serious for vector and scalar modes when choosing Lorentz-violating mass term [26]. We will confirm that the Hořava scalar survives in the massless limit of Fierz-Pauli mass term (vDVZ discontinuity), but it is absent in the massless limit of Lorentz-violating mass term (no vDVZ discontinuity) [7].

2 dHL gravity

First of all, we introduce the ADM formalism where the metric is parameterized as

$$ds_{ADM}^2 = -N^2 dt^2 + g_{ij} (dx^i - N^i dt) (dx^j - N^j dt), \quad (1)$$

Then, the Einstein-Hilbert action can be expressed as

$$S^{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g} N (K_{ij} K^{ij} - K^2 + R), \quad (2)$$

where G is Newton's constant and extrinsic curvature K_{ij} takes the form

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i). \quad (3)$$

Here, a dot denotes a derivative with respect to t (" \cdot " = $\frac{\partial}{\partial t}$).

On the other hand, the action of the dHL gravity is given by [6]

$$S^{dHL} = \int dt d^3\mathbf{x} (\mathcal{L}^0 + \sqrt{g} N \mu^4 R + \mathcal{L}^h), \quad (4)$$

$$\mathcal{L}^0 = \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W R - 3\Lambda_W^2)}{8(1-3\lambda)} \right\}, \quad (5)$$

$$\mathcal{L}^h = \sqrt{g} N \left\{ \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)} R^2 - \frac{\kappa^2}{\eta^4} \left(C_{ij} - \frac{\mu \eta^2}{2} R_{ij} \right) \left(C^{ij} - \frac{\mu \eta^2}{2} R^{ij} \right) \right\}. \quad (6)$$

Here C_{ij} is the Cotton tensor defined by

$$C^{ij} = \epsilon^{ik\ell} \nabla_k \left(R^j_\ell - \frac{1}{4} R \delta^j_\ell \right) \quad (7)$$

which is obtained from the variation of gravitational Chern-Simons term with coupling $1/\eta^2$. The full equations of motion were derived in [27] and [28], but we do not write them due to the length. Taking a limit of $\Lambda_W \rightarrow 0$ in $\mathcal{L}^0 + \sqrt{g}N\mu^4 R$, we obtain the λR -model [6]

$$S^{\lambda R} \equiv \int dt d^3x \tilde{\mathcal{L}}^{\lambda R} = \int dt d^3x \sqrt{g}N \left[\frac{2}{\kappa^2} (K_{ij}K^{ij} - \lambda K^2) + \mu^4 R \right]. \quad (8)$$

Comparing Eq.(8) with general relativity (2), the speed of light and Newton's constant are determined by

$$c^2 = \frac{\kappa^2 \mu^4}{2}, \quad G = \frac{\kappa^2}{32\pi c}, \quad \lambda = 1. \quad (9)$$

Since we consider the $z = 3$ Hořava-Lifshitz gravity, scaling dimensions are $[t] = -3$, $[x] = -1$, $[\kappa] = 0$, $[\mu] = 1$, and $[c] = 2$. Even though the scaling dimensions are relevant to the UV properties, these are also necessary to define the linearized theory of $z = 3$ Hořava-Lifshitz gravity consistently. The reason is that we have to keep the same dimensions six for all terms, although couplings of the kinetic term ($2/\kappa^2$) and the sixth order derivatives ($\kappa^2/2\eta^4$) are dimensionless. In order to see the UV properties of power-counting renormalizability, it is better to switch from the $c = 1$ units to (9) units that impose the scaling dimensions. Switching back to $c = 1$ units leads to the case [17]

$$S_{IR}^{\lambda R} = \mu^4 \int dt d^3x \sqrt{g}N \left[(K_{ij}K^{ij} - \lambda K^2) + R \right] \quad (10)$$

which is suitable for discussing the IR properties (large distances) of strong coupling problem and vDVZ discontinuity.

The deformed Lagrangian which is relevant to our study takes the form [6]

$$\tilde{\mathcal{L}} \equiv \tilde{\mathcal{L}}^{\lambda R} + \mathcal{L}^h \quad (11)$$

$$= \sqrt{g}N \left[\frac{2}{\kappa^2} (K_{ij}K^{ij} - \lambda K^2) + \mu^4 \left(R + \frac{1}{2\omega} \frac{4\lambda - 1}{3\lambda - 1} R^2 - \frac{2}{\omega} R_{ij}R^{ij} \right) \right] \quad (12)$$

$$+ \frac{\kappa^2 \mu}{2\eta^2} \epsilon^{ijk} R_{il} \nabla_j R_k^l - \frac{\kappa^2}{2\eta^4} C_{ij}C^{ij} \quad (13)$$

where a characterized parameter ω is given by

$$\omega = \frac{16\mu^2}{\kappa^2} = \frac{16\sqrt{2}c}{\kappa^3}. \quad (14)$$

Actually, the Lagrangian (12) is enough to describe scalar and vector propagations because (13) from the Cotton tensor contributes to tensor propagations only. For $\lambda = 1$, taking the limit of $\omega \rightarrow \infty$ while keeping $c^2 = 1$ fixed is equivalent to recovering the Einstein gravity ($\lambda = 1R$ -model). Explicitly, this limit implies $\kappa^2 \rightarrow 0$ ($\mu^4 \sim \kappa^{-2} \rightarrow \infty$) which means that

the kinetic term and curvature term $\mu^4 R$ dominate over all higher order curvature terms. The deformed Lagrangian (11) can be redefined to be

$$\tilde{\mathcal{L}} = \mathcal{L}_K + \mathcal{L}_V, \quad (15)$$

where $\mathcal{L}_K(\mathcal{L}_V)$ denote the kinetic (potential) Lagrangian with (without) temporal derivative terms.

We wish to consider perturbations of the metric around Minkowski spacetimes, which is a solution to the full Lagrangian (11)

$$g_{ij} = \delta_{ij} + \eta h_{ij}, \quad N = 1 + \eta n, \quad N_i = \eta n_i, \quad (16)$$

where a dimensionless coupling constant η from gravitational Chern-Simons term is included to define the perturbation. The inclusion of η makes sense because the non-interacting limit corresponds to sending $\eta \rightarrow 0$, while keeping the ratio $\gamma = \kappa/\eta$ fixed [1]. This in turn provides the limit of $\kappa \rightarrow 0$ ($\omega \rightarrow \infty$). For $\lambda = 1$, this limit yields a one-parameter family of free-field fixed points parameterized by γ .

At quadratic order the λR -action (8) turns out to be

$$\begin{aligned} S_2^{\lambda R} = \eta^2 \int dt d^3x & \left\{ \frac{1}{\kappa^2} \left[\frac{1}{2} \dot{h}_{ij}^2 - \frac{\lambda}{2} \dot{h}^2 + (\partial_i n_j)^2 + (1 - 2\lambda)(\partial \cdot n)^2 - 2\partial_i n_j (\dot{h}_{ij} - \lambda \dot{h} \delta_{ij}) \right] \right. \\ & \left. + \frac{\mu^4}{2} \left[-\frac{1}{2} (\partial_k h_{ij})^2 + \frac{1}{2} (\partial_i h)^2 + (\partial_i h_{ij})^2 - \partial_i h_{ij} \partial_j h + 2n(\partial_i \partial_j h_{ij} - \partial^2 h) \right] \right\} \quad (17) \end{aligned}$$

with $h = h_{ii}$. A general Lorentz-violating mass term is given by [26]

$$S_2^{LV} = \frac{\eta^2}{2\kappa^2} \int dt d^3x \left[4m_0^2 n^2 + 2m_1^2 n_i^2 - \tilde{m}_2^2 h_{ij}^2 + \tilde{m}_3^2 h^2 + 4\tilde{m}_4^2 n h \right]. \quad (18)$$

As was pointed out in [29], S_2^{LV} provides various phases of massive gravity in general relativity. In this work, we add (18) to the linearized theory of dHL gravity to investigate strong coupling problem and the vDVZ discontinuity. In this work, we choose the case of $m_0 = 0$, where the lapse field n enters the action linearly and thus, it still acts as a Lagrange multiplier. If one considers a non-zero mass m_0 seriously, it induces a ghost instability [26, 30]. At this stage, we would like to mention that for generic backgrounds, $m_1^2 = 0$ has provided a well-defined case in bi-gravity and massive gravity [31, 30]. Also, the generic case could be well behaved in generic backgrounds [31].

We compare (18) with the Lorentz-invariant Fierz-Pauli mass term [32]

$$S_2^{FP} = \frac{\eta^2}{2\kappa^2} \int dt d^3x \left[-m^2 h_{\mu\nu} h^{\mu\nu} + m^2 \left(h^\mu{}_\mu \right)^2 \right]. \quad (19)$$

In order to analyze physical propagations thoroughly, it is convenient to use the cosmological decomposition in terms of scalar, vector, and tensor modes under spatial rotations $SO(3)$ [33]

$$\begin{aligned} n &= -\frac{1}{2}A, \\ n_i &= (\partial_i B + V_i), \\ h_{ij} &= (\psi \delta_{ij} + \partial_i \partial_j E + 2\partial_{(i} F_{j)} + t_{ij}), \end{aligned} \quad (20)$$

where the conditions of $\partial^i F_i = \partial^i V_i = \partial^i t_{ij} = t_{ii} = 0$ are imposed. The last two conditions mean that t_{ij} is a transverse and traceless tensor in three spatial dimensions. Using this decomposition, the scalar modes (A, B, ψ, E) , the vector modes (V_i, F_i) , and the tensor modes (t_{ij}) decouple completely from each other. These all amount to 10 degrees of freedom for a symmetric tensor in four dimensions.

Before proceeding, let us check dimensions. Masses have scaling dimensions: $[m_1^2] = 2$ and $[\tilde{m}_2^2] = [\tilde{m}_3^2] = [\tilde{m}_4^2] = 6$. In order to get the true mass with dimension 1, we redefine mass squares as

$$\tilde{m}_i^2 = c^2 m_i^2, \quad \text{for } i = 2, 3, 4 \quad (21)$$

which implies that $[m_i^2] = 2$. The Fierz-Pauli mass term is recovered when all masses are equal except for m_0 as

$$m_1^2 = m_2^2 = m_3^2 = m_4^2 = m^2; \quad m_0 = 0. \quad (22)$$

The quadratic action for λR -model is obtained by substituting (20) into the quadratic action (17) as

$$\begin{aligned} S_2^{\lambda R} = \frac{1}{2\gamma^2} \int dt d^3x \left\{ \right. & \left[3(1-3\lambda)\dot{\psi}^2 + 2\partial_i w_j \partial^i w^j - 4\left((1-3\lambda)\dot{\psi} + (1-\lambda)\partial^2 \dot{E}\right)\partial^2 B \right. \\ & \left. + 4(1-\lambda)(\partial^2 B)^2 + 2(1-3\lambda)\dot{\psi}\partial^2 \dot{E} + (1-\lambda)(\partial^2 \dot{E})^2 + t_{ij}\dot{t}^{ij} \right] \\ & \left. + c^2 \left(2\partial_k \psi \partial^k \psi + 4A\partial^2 \psi - \partial_k t_{ij} \partial^k t^{ij} \right) \right\} \quad (23) \end{aligned}$$

with $\gamma^2 = \kappa^2/\eta^2$ and $w_i = V_i - \dot{F}_i$. We have the coupling of $\frac{1}{2\gamma^2}$ in the quadratic action. The higher order action from \mathcal{L}^h takes the form

$$S_2^h = \frac{\kappa^2 \mu^2 \eta^2}{8} \int dt d^3x \left[-\frac{1-\lambda}{2(1-3\lambda)} \psi \partial^4 \psi - \frac{1}{4} t_{ij} \partial^4 t^{ij} + \frac{1}{\mu \eta^2} \epsilon^{ijk} t_{il} \partial^4 \partial_j t^l{}_k + \frac{1}{\mu^2 \eta^4} t_{ij} \partial^6 t^{ij} \right]. \quad (24)$$

We find that two modes of scalar ψ and tensor t_{ij} exist in S_2^h only, missing vector modes. Since the spatial slice is conformally flat, the vanishing Cotton tensor and the absence of six derivative terms result in the scalar sector. Also, the Cotton tensor does not contribute

to vector modes (V_i, \dot{F}_i) . The vectors are absent in S_2^h because the vector belongs to gauge degrees of freedom in the massless gravity theory.

Before we proceed, we mention the foliation-preserving diffeomorphism (FDiff) in the dHL gravity with the projectability condition. Considering the anisotropic scaling of temporal and spatial coordinates $(t \rightarrow b^z t, x^i \rightarrow b x^i)$, the time coordinate t plays a privileged role. A quadratic action of $S_2^{\lambda R} + S_2^h$ should be invariant under FDiff whose transformation is given by

$$t \rightarrow \tilde{t} = t + \epsilon^0(t), \quad x^i \rightarrow \tilde{x}^i = x^i + \epsilon^i(t, \mathbf{x}), \quad (25)$$

which shows that the spacetime symmetry is smaller than the full diffeomorphism (Diff) in the general relativity

$$t \rightarrow \tilde{t} = t + \epsilon^0(t, \mathbf{x}), \quad x^i \rightarrow \tilde{x}^i = x^i + \epsilon^i(t, \mathbf{x}). \quad (26)$$

FDiff (Diff) invariance are dynamical symmetry of dHL gravity with the projectability condition (general relativity) and not just symmetry of the background spacetimes. Hence, it controls the number of propagating degrees of freedom: more symmetry means less degrees of freedom. It is well known that general relativity as a massless gravity theory has two degrees of freedom, while the dHL gravity with the projectability condition may have three.

In this work, we consider the dHL gravity without imposing the projectability condition. In this case, Diff is more suitable than FDiff. Using the notation of $\epsilon^\mu = (\epsilon^0, \epsilon^i)$ and $\epsilon_\nu = \eta_{\nu\mu} \epsilon^\mu$, the perturbation of metric transforms as

$$\delta g_{\mu\nu} \rightarrow \delta \tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \quad (27)$$

Further, making a decomposition ϵ^i into a scalar ξ and a pure vector ζ^i as $\epsilon^i = \partial^i \xi + \zeta^i$ with $\partial_i \zeta^i = 0$, one finds the transformation for scalars

$$A \rightarrow \tilde{A} = A - 2\dot{\epsilon}^0, \quad \psi \rightarrow \tilde{\psi} = \psi, \quad B \rightarrow \tilde{B} = B - \epsilon^0 + \dot{\xi}, \quad E \rightarrow \tilde{E} = E + 2\xi. \quad (28)$$

On the other hand, the vector and the tensor take the forms

$$V_i \rightarrow \tilde{V}_i = V_i + \dot{\zeta}_i, \quad F_i \rightarrow \tilde{F}_i = F_i + \zeta_i, \quad t_{ij} \rightarrow \tilde{t}_{ij} = t_{ij}. \quad (29)$$

For the Diff transformations, gauge invariant combinations are

$$t_{ij}, \quad w_i = V_i - \dot{F}_i, \quad (30)$$

for tensor and vector, respectively and

$$\psi, \quad \Phi = c^2 A - \dot{\Pi} \quad \text{with} \quad \Pi = 2B - \dot{E} \quad (31)$$

for two scalar modes². We note that Π is not a gauge-invariant scalar mode.

Let us try to express the quadratic action (23) in terms of gauge-invariant quantities as [9]

$$S_2^{\lambda R} = \frac{1}{2\gamma^2} \int dt d^3x \left\{ \left[3(1-3\lambda)\dot{\psi}^2 - 2w_i \triangle w^i - 2(1-3\lambda)\dot{\psi} \triangle \Pi + (1-\lambda)(\triangle \Pi)^2 \right. \right. \\ \left. \left. + \dot{t}_{ij}\dot{t}^{ij} \right] + c^2 \left(-2\psi \triangle \psi + 4 \triangle A\psi + t_{ij} \triangle t^{ij} \right) \right\} \quad (32)$$

with the spatial Laplacian $\triangle = \partial^2$. However, it is no doubt that for general λ , the quadratic action of λR -model is not expressed in terms of gauge-invariant quantities. This contrasts to the Hamiltonian approach which shows that the value of λ is completely irrelevant for finding two physical degrees of freedom for a massless graviton [17]. In the Hamiltonian approach, they have used Diff as full dynamical symmetry and have chosen a gauge-fixing to identify four degrees of freedom in phase space. However, in this Lagrangian approach, we are working with the quadratic action and not making a gauge-fixing. The Diff could be manifestly realized at the quadratic action only for $\lambda = 1$. Then, the $\lambda = 1$ case leads to the gauge-invariant action as

$$S_2^{\lambda=1R} = \frac{1}{2\gamma^2} \int dt d^3x \left\{ \left[-6\dot{\psi}^2 - 2w_i \triangle w^i + 4\psi \triangle \Phi + \dot{t}_{ij}\dot{t}^{ij} \right] + c^2 \left[-2\psi \triangle \psi + t_{ij} \triangle t^{ij} \right] \right\}. \quad (33)$$

At this stage, it is unclear why the value of λ is uniquely determined to be 1 in the perturbation theory. Other possibility includes the case that for generic λ , Π and A are separately gauge-invariant scalars. However, this is not the case. An allowable case is that for generic λ , Π is a gauge-invariant scalar and A is a parameter, which is exactly the dHL gravity with the projectability condition. We note that S_2^h in (24) contains only ψ and t_{ij} , which are gauge-invariant quantities.

On the other hand, the mass term (18) leads to

$$S_2^{LV} = \frac{1}{2\gamma^2} \int dt d^3x \left[2 m_1^2 (V_i^2 + (\partial_i B)^2) \right. \\ \left. - \tilde{m}_2^2 (t_{ij}\dot{t}^{ij} + 2(\partial_i F_j)^2 + (\partial_i \partial_j E)^2 + 2\psi \partial^2 E + 3\psi^2) \right. \\ \left. + \tilde{m}_3^2 (\partial^2 E + 3\psi)^2 - 2\tilde{m}_4^2 A (\partial^2 E + 3\psi) \right]. \quad (34)$$

which is not obviously invariant under Diff because we could not express whole terms in terms of gauge-invariant quantities. However, these do not give rise to any problem because we are interested in the massless limit of Lorentz-violating mass term and we do not impose any gauge to perform the perturbation analysis around the Minkowski background.

²For TDiff respecting an additional constraint $\partial_\mu \epsilon^\mu = 0$ [34], there are three gauge-invariant scalar modes: ψ , Φ , and $\Theta = A - \triangle E$. In this case, a truly propagating scalar graviton is given by ψ .

3 Massive propagations

Without the projectability condition, we conjecture that out of the 5 DOF of a massive graviton, 2 of these are expressed as transverse and traceless tensor modes t_{ij} , 2 of these are expressed as transverse vector modes F_i , and the remaining one is from Hořava scalar ψ .

3.1 Tensor modes

The field equation for tensors is given by

$$\ddot{t}_{ij} - c^2 \Delta t_{ij} + c^2 m_2^2 t_{ij} + \frac{2c^2}{\omega} \Delta^2 t_{ij} - \frac{\kappa^4 \mu}{4\eta^2} \epsilon_{ilm} \partial^l \Delta^2 t_j{}^m - \frac{\kappa^4}{4\eta^4} \Delta^3 t_{ij} = 0. \quad (35)$$

The requirement that these modes are not tachyonic gives the stability condition

$$m_2^2 \geq 0. \quad (36)$$

In the absence of mass, these modes describe the chiral primordial gravitational waves [12, 35]. These circularly polarized modes are possible because the Cotton tensor C_{ij} is present, making parity violation. In the presence of mass term, it may describe massive chiral gravitational waves.

3.2 Vector modes

It is clear from Eqs.(23) and (34) that V_i enters the action without temporal derivatives, that is, it is a non-dynamical field in the massless theory. A massive vector Lagrangian takes the form

$$\mathcal{L}^v = \frac{1}{\gamma^2} \left[-w_i \Delta w^i + m_1^2 V_i^2 - \tilde{m}_2^2 (\partial_i F_j)^2 \right] \quad (37)$$

with $w_i = V_i - \dot{F}_i$. It is obvious that in the absence of mass terms, w_i is a non-propagating vector mode. We integrate V_i out using the field equation obtained by varying the action with respect to V_i

$$\Delta (V_i - \dot{F}_i) - m_1^2 V_i = 0 \quad (38)$$

which implies

$$V_i = \frac{\Delta}{\Delta - m_1^2} \dot{F}_i. \quad (39)$$

Plugging this expression into Eq.(37) leads to be

$$\mathcal{L}^v = \frac{1}{\gamma^2} \left[\frac{\Delta m_1^2}{\Delta - m_1^2} \dot{F}_i^2 + \tilde{m}_2^2 F_i \Delta F^i \right]. \quad (40)$$

In order to obtain a canonical action, we introduce a canonical vector field \tilde{F}_i defined by

$$F_i = \frac{\gamma}{m_1} \sqrt{\frac{\Delta - m_1^2}{2\Delta}} \tilde{F}_i \propto \frac{1}{m_1 M_{Pl}} \sqrt{\frac{\Delta - m_1^2}{2\Delta}} \tilde{F}_i \quad (41)$$

in the $c = 1$ units. Then, the Lagrangian (40) takes a canonical form

$$\mathcal{L}_c^v = \frac{1}{2} \left[\dot{\tilde{F}}_i^2 - \frac{m_2^2}{m_1^2} (\partial_i \tilde{F}_j)^2 - m_2^2 \tilde{F}_j^2 \right]. \quad (42)$$

Now let us discuss the strong coupling issue. In order to discuss the strong coupling problem, we first note that

$$\frac{1}{8\pi G} = \frac{4c}{\kappa^2} \equiv M_{Pl}^2, \quad (43)$$

which leads to a relation between γ and Planck mass scale M_{Pl}

$$\gamma = \frac{2\sqrt{c}}{\eta M_{Pl}} \propto \frac{1}{M_{Pl}} \quad (44)$$

in the $c = 1$ units. Considering the relation Eq.(41), the original vector field is proportional to $(m M_{Pl})^{-1}$ and from Eq.(37), a gauge-invariant combination w_i takes the form

$$w_i \propto \frac{m}{M_{Pl}} \tilde{F}_i \quad (45)$$

which shows that vector modes at small m is precisely the same as in the Fierz-Pauli case. The analysis in Ref.[36] suggests that the strong coupling occurs at $E \sim \sqrt{m M_{Pl}}$, which is a high scale. In comparison to the Fierz-Pauli case, vector field changes nothing except the speed of light. Its equation of motion is given by

$$\ddot{\tilde{F}}_i - \frac{m_2^2}{m_1^2} \Delta \tilde{F}_j + m_2^2 \tilde{F}_i = 0, \quad (46)$$

which leads to the dispersion relation

$$\omega^2 = \frac{m_2^2}{m_1^2} k^2 + m_2^2. \quad (47)$$

For $m_1^2 > 0$ and $m_2^2 > 0$, it is obvious that there is no ghosts.

In the Fierz-Pauli case of $m_1^2 = m_2^2$, the massive vector equation reduces to

$$(\square - m^2) \tilde{F}_i = 0 \quad (48)$$

which represents a massive vector with two degrees of freedom. Here $\square = -\partial_0^2 + \Delta$ with $\partial_0 = \frac{\partial}{\partial x^0}$ with $x^0 = ct$.

3.3 Scalar mode

The scalar Lagrangian with four different masses takes the form

$$\begin{aligned} \mathcal{L}_\lambda^s = \frac{1}{2\gamma^2} \left[-3(3\lambda - 1)\dot{\psi}^2 + 2(3\lambda - 1)\dot{\psi} \triangle \Pi - (\lambda - 1)(\triangle \Pi)^2 - \frac{\kappa^4 \mu^2 (1 - \lambda)}{8(3\lambda - 1)} \psi \triangle^2 \psi \right. \\ \left. + \mu^4 \kappa^2 \left(-\psi \triangle \psi + 2A \triangle \psi \right) - 2m_1^2 B \triangle B - \tilde{m}_2^2 \left(E \triangle^2 E + 2\psi \triangle E + 3\psi^2 \right) \right. \\ \left. + \tilde{m}_3^2 \left(\triangle E + 3\psi \right)^2 - 2\tilde{m}_4^2 A \left(\triangle E + 3\psi \right) \right]. \end{aligned} \quad (49)$$

From the Lagrangian (49), we find that there exist a Lagrange multiplier A and a non-dynamical field B . Their variations with respect to A are given by

$$\triangle \psi - \frac{\tilde{m}_4^2}{c^2} \left(\triangle E + 3\psi \right) = 0 \quad (50)$$

which implies that E can be expressed in terms of ψ

$$E = \frac{2c^2 \psi}{\tilde{m}_4^2} - 3 \frac{\psi}{\triangle}. \quad (51)$$

On the other hand, the variation with respect to B leads to

$$(3\lambda - 1)\dot{\psi} + (\lambda - 1) \triangle \dot{E} - 2(\lambda - 1) \triangle B - m_1^2 B = 0, \quad (52)$$

Using Eqs.(51) and (52), we can express B in terms of ψ as

$$B = \frac{2}{2(\lambda - 1) \triangle + m_1^2} \left[(\lambda - 1) \frac{2c^2 \triangle \dot{\psi}}{\tilde{m}_4^2} + \dot{\psi} \right]. \quad (53)$$

Hence, we rewrite Π in terms of Hořava scalar ψ as

$$\Pi = 2B - \dot{E} = \frac{4}{2(\lambda - 1) \triangle + m_1^2} \left[(\lambda - 1) \frac{2c^2 \triangle \dot{\psi}}{\tilde{m}_4^2} + \dot{\psi} \right] - \frac{2c^2 \dot{\psi}}{\tilde{m}_4^2} + \frac{3}{\triangle} \dot{\psi}. \quad (54)$$

Plugging E , B , and Π into (49) leads to a very complicated Lagrangian for ψ

$$\begin{aligned} \mathcal{L}_\lambda^\psi = \frac{1}{2\gamma^2} \left[-3(3\lambda - 1)\ddot{\psi}\psi + 4(3\lambda - 1) \frac{\ddot{\psi} \triangle \psi}{m_4^2} - \frac{\kappa^4 \mu^2 (1 - \lambda)}{8(3\lambda - 1)} \psi \triangle^2 \psi \right. \\ \left. - \frac{(3\lambda - 1)}{2(\lambda - 1) \triangle + m_1^2} \left(8(\lambda - 1) \frac{\ddot{\psi} \triangle^2 \psi}{m_4^2} + 8\ddot{\psi} \triangle \psi \right) \right. \\ \left. + \frac{2(\lambda - 1)}{2(\lambda - 1) \triangle + m_1^2} \left(-8(\lambda - 1) \frac{\ddot{\psi} \triangle^3 \psi}{m_4^4} + [12(\lambda - 1) - 8] \frac{\ddot{\psi} \triangle^2 \psi}{m_4^2} + 12\ddot{\psi} \triangle \psi \right) \right. \\ \left. + \frac{16(\lambda - 1) + \frac{8m_1^2}{\triangle}}{[2(\lambda - 1) \triangle + m_1^2]^2} \left((\lambda - 1)^2 \frac{\ddot{\psi} \triangle^4 \psi}{m_4^4} + 2(\lambda - 1) \frac{\ddot{\psi} \triangle^3 \psi}{m_4^2} + \ddot{\psi} \triangle^2 \psi \right) \right. \\ \left. + (\lambda - 1) \left(\frac{4\ddot{\psi} \triangle^2 \psi}{m_4^4} - \frac{12\ddot{\psi} \triangle \psi}{m_4^2} + 9\ddot{\psi}\psi \right) \right. \\ \left. - 4 \left(\frac{m_2^2 - m_3^2}{m_4^2} \right) \psi (c^2 \triangle^2) \psi + \left(8 \frac{m_2^2}{m_4^2} - 2 \right) \psi c^2 \triangle \psi - 6c^2 m_2^2 \psi^2 \right]. \end{aligned} \quad (55)$$

It seems difficult to derive the equation of motion for the Hořava scalar. However, in the limit of $\lambda \rightarrow 1$, we have a simplified Lagrangian for ψ

$$\begin{aligned} \mathcal{L}_{\lambda=1}^{\psi} = \frac{1}{2\gamma^2} & \left[-6\ddot{\psi}\psi + \left(\frac{8}{m_4^2} - \frac{8}{m_1^2} \right) \ddot{\psi} \triangle \psi - 4 \left(\frac{m_2^2 - m_3^2}{m_4^4} \right) \psi c^2 \triangle^2 \psi \right. \\ & \left. + \left(8 \frac{m_2^2}{m_4^2} - 2 \right) \psi c^2 \triangle \psi - 6c^2 m_2^2 \psi^2 \right], \end{aligned} \quad (56)$$

which is identified with the quadratic Lagrangian for general relativity with the LV mass term [26]. The equation of motion for ψ is given by

$$\begin{aligned} 6\ddot{\psi} - 8 \left(\frac{1}{m_4^2} - \frac{1}{m_1^2} \right) \triangle \ddot{\psi} \\ - 2 \left(\frac{4m_2^2}{m_4^2} - 1 \right) c^2 \triangle \psi + 4c^2 \left(\frac{m_2^2 - m_3^2}{m_4^4} \right) \triangle^2 \psi + 6c^2 m_2^2 \psi = 0. \end{aligned} \quad (57)$$

For $m_1^2 > m_4^2, m_2^2 > m_3^2, 4m_2^2 > m_4^2$, there are no ghost because all terms in the equation have correct signs (\triangle is negative-definite). This contrasts to that of dHL gravity with the projectability condition [19], which indicates that there is no ghost free, massive propagation for the Hořava scalar ψ .

In order to discuss the strong coupling issue, let us remind the relation of $\gamma \propto 1/M_{Pl}$ in the $c = 1$ units. Considering the Lagrangian (56), we could define the normalization factor relating ψ and its canonically normalized field ψ^c as

$$\psi = \gamma \left[-8 \left(\frac{1}{m_4^2} - \frac{1}{m_1^2} \right) \triangle + 6 \right]^{-1} \psi^c \propto \frac{m}{M_{Pl}} \psi^c. \quad (58)$$

Also, considering Eqs.(51) and (53), one finds that

$$E, B \propto \frac{1}{m M_{Pl}} \psi^c \quad (59)$$

which are in a complete analogy to vector modes. All these ensure that the strong coupling scale in the scalar sector is the same as in vector sector ($E \sim \sqrt{m M_{Pl}}$).

However, in the case of Fierz-Pauli mass (22), this picture is changed. In this case, we have

$$\psi = \gamma \psi^c \quad (60)$$

and

$$E, B \propto \frac{1}{m^2 M_{Pl}} \psi^c \quad (61)$$

which implies the low energy scale of strong coupling ($E \sim (m^4 M_{Pl})^{1/5}$) [36]. Moreover, the equation (57) reduces to a simpler equation

$$\ddot{\psi} - c^2 \triangle \psi + c^2 m^2 \psi = 0 \quad (62)$$

which is nothing but the massive Klein-Gordon equation

$$(\square - m^2)\psi = 0. \quad (63)$$

Finally, we confirm that a massive graviton takes five degrees of freedom for both the Lorentz-violating and Fierz-Pauli mass terms.

4 vDVZ discontinuity

Since the Lagrangian (55) takes a complicated form, it is a formidable task to prove that for generic λ , there is no vDVZ discontinuity in the massless limit. Instead, we will show that for $\lambda = 1$ case, there is no the vDVZ discontinuity, in comparison to the Fierz-Pauli case. In order to show it, we have to introduce the external source term

$$S_{int} = -\frac{1}{\gamma^2} \int dt d^3x \left[h^{ij} T_{ij} + 2h^{0j} T_{0j} + h^{00} T_{00} \right]. \quad (64)$$

A covariant form of the source conservation-law $\partial_\mu T^{\mu\nu} = 0$ is slightly modified to have

$$\dot{T}_{00} = a \partial_j T_{j0}, \quad \dot{T}_{0i} = \partial_j T_{ji} \quad (65)$$

where a with $[a] = 4$ is inserted to have correct scaling dimensions

$$[T_{ij}] = 6, \quad [T_{0j}] = 4, \quad [T_{00}] = 6. \quad (66)$$

On later, a will be determined to be $a = c^2$. Then, we could express the above in terms of gauge-invariant modes as

$$S_{int} = -\frac{1}{\gamma^2} \int dt d^3x \left[t_{ij} T_{ij} - 2w_i T_{0i} + (aA - \dot{\Pi}) \frac{T_{00}}{a} + \psi T_{ii} \right]. \quad (67)$$

Choosing $a = c^2$, we note that $c^2 A - \dot{\Pi}$ is nothing but a gauge-invariant scalar Φ under Diff transformations. We wish to study the vDVZ discontinuity by making use of (67).

4.1 $\lambda = 1R$ -model

First of all, we consider the quadratic action of $S_2^{\lambda=1R}$ in (32) together with external source in (67) to find massless propagations in general relativity. Using

$$\triangle \rightarrow -\mathbf{k} \cdot \mathbf{k} = -k^2, \quad \ddot{t}_{ij} \rightarrow -\varpi^2 t_{ij}, \quad (68)$$

propagators with source are derived as [26]

$$\ddot{t}_{ij} - \triangle t_{ij} = -T_{ij} \rightarrow t_{ij}(k) = \frac{T_{ij}}{\varpi^2 - k^2}, \quad (69)$$

$$w_i = \frac{T_{0i}}{\Delta} \rightarrow w_i(k) = -\frac{T_{0i}}{k^2}, \quad (70)$$

$$\psi = \frac{T_{00}}{2\Delta} \rightarrow \psi(k) = -\frac{T_{00}}{2k^2}, \quad (71)$$

$$\Phi = \frac{1}{2\Delta} \left[T_{ii} + T_{00} - 3\frac{\ddot{T}_{00}}{\Delta} \right] \rightarrow \Phi(k) = -\frac{1}{2k^2} \left[T_{ii} + T_{00} - \frac{3\varpi^2 T_{00}}{k^2} \right], \quad (72)$$

where $\Phi = A - \ddot{\Pi}$ is a gauge-invariant scalar which plays the role of Newtonian potential. The above shows clearly that tensor modes t_{ij} are propagating on the Minkowski background, while vector and all scalars are non-propagating because there is no kinetic terms ϖ^2 (second order temporal derivative terms). It confirms that the $\lambda = 1R$ -model has two propagating degrees of freedom for a massless graviton, which is equivalent to the general relativity. All propagators (69)-(72) are those of the dHL gravity in the limit of $\omega \rightarrow \infty$ and in the massless limit.

4.2 Tensor modes

The tensor equation takes a relatively simple form

$$\ddot{t}_{ij} - c^2 \Delta t_{ij} + c^2 m_2^2 t_{ij} + \frac{2c^2}{\omega} \Delta^2 t_{ij} - \frac{\kappa^4 \mu}{4\eta^2} \epsilon_{ilm} \partial^l \Delta^2 t_j{}^m - \frac{\kappa^4}{4\eta^4} \Delta^3 t_{ij} = -T_{ij}. \quad (73)$$

We find that there is no the vDVZ discontinuity because a single mass term $m_2^2 (\geq 0)$ is present. It is obvious that for $T_{ij} = 0$, the above equation reduces to Eq.(35). In the massless limit of $m_2^2 \rightarrow 0$, Eq.(73) has described chiral gravitational waves without ghost propagating on the Minkowski background [35]. In the massive case, (73) may describe massive chiral gravitational waves.

4.3 Vector modes

From (37) and (67), the vector equations are derived as

$$\Delta(V_i - \dot{F}_i) - m_1^2 V_i = T_{0i}, \quad (74)$$

$$\dot{V}_i - \ddot{F}_i - \tilde{m}_2 F_i = \frac{1}{\Delta} \dot{T}_{0i}. \quad (75)$$

From equation (74), we find

$$V_i = \frac{\Delta}{\Delta - m_1^2} \dot{F}_i + \frac{1}{\Delta - m_1^2} T_{0i}. \quad (76)$$

Plugging this into equation (75) leads to

$$F_i = -\frac{1}{\Delta \left(\partial_t^2 - c^2 \frac{m_2^2}{m_1^2} \Delta + c^2 m_2^2 \right)} \dot{T}_{0i} = -\frac{1}{c^2 \Delta \left(\partial_0^2 - \frac{m_2^2}{m_1^2} \Delta + m_2^2 \right)} \dot{T}_{0i}. \quad (77)$$

We define the massless limit of the Lorentz-violating mass term as [26]

$$\text{MLLV} : \quad m_i^2 \rightarrow 0, \quad \frac{m_i^2}{m_j^2} \rightarrow \text{fixed}, \quad i, j = 1, \dots, 4, \quad (78)$$

while the massless limit of the Lorentz-invariant Fierz-Pauli mass term is defined by

$$\text{MLFP} : \quad m_i^2 = m^2 \rightarrow 0, \quad i, j = 1, \dots, 4. \quad (79)$$

The gauge-invariant vector takes the form

$$\begin{aligned} w_i &= V_i - \dot{F}_i = \frac{m_1^2}{\Delta - m_1^2} \dot{F}_i + \frac{1}{\Delta - m_1^2} T_{0i}, \\ &= -\frac{m_1^2}{\Delta - m_1^2} \frac{1}{c^2 \Delta \left(\partial_0^2 - \frac{m_2^2}{m_1^2} \Delta + m_2^2 \right)} \ddot{T}_{0i} + \frac{1}{\Delta - m_1^2} T_{0i}. \end{aligned} \quad (80)$$

Under the MLLV of (78), the gauge-invariant vector reduces to (70)

$$w_i = \frac{1}{\Delta} T_{0i}, \quad (81)$$

which shows that the vector is non-propagating in the massless limit. Also, we note that nothing changes for the Fierz-Pauli case of $m_1^2 = m_2^2$ because the gauge-invariant vector takes the form

$$w_i = -\frac{m^2}{\Delta - m^2} \frac{1}{c^2 \Delta \left(\partial_0^2 - \Delta + m^2 \right)} \ddot{T}_{0i} + \frac{1}{\Delta - m^2} T_{0i}, \quad (82)$$

which leads again to Eq.(81) in the MLFP of Eq.(79).

4.4 Scalar modes with $\lambda = 1$

In the scalar sector, the field equations are obtained by variation of (49)+(67) with respect to A, B, E , and ψ as

$$2 \Delta \psi - \frac{\tilde{m}_4^2}{c^2} (\Delta E + 3\psi) = \frac{T_{00}}{c^2}, \quad (83)$$

$$2\dot{\psi} - m_1^2 B = \frac{1}{\Delta} \frac{\dot{T}_{00}}{a}, \quad (84)$$

$$2\ddot{\psi} - \tilde{m}_2^2 (\Delta E + \psi) + \tilde{m}_3^2 (\Delta E + 3\psi) - \tilde{m}_4^2 A = \frac{1}{\Delta} \frac{\ddot{T}_{00}}{a}, \quad (85)$$

$$2 \Delta \Phi - 2c^2 \Delta \psi + 2\tilde{m}_2^2 \Delta E + \frac{\kappa^4 \mu^2}{8} \Delta^2 \psi = t_{ii} - \frac{3}{\Delta} \frac{\ddot{T}_{00}}{a}, \quad (86)$$

where $\Phi = c^2 A - 2\dot{B} + \ddot{E}$ is the Newtonian potential [26, 30]. Eq.(83) provides

$$E = \frac{2c^2}{\tilde{m}_4^2} \psi - \frac{3}{\Delta} \psi - \frac{T_{00}}{\tilde{m}_4^2 \Delta}, \quad (87)$$

while Eq.(84) gives

$$B = \frac{2\dot{\psi}}{m_1^2} - \frac{1}{m_1^2 \Delta} \frac{\dot{T}_{00}}{a}. \quad (88)$$

Eq.(85) leads to

$$A = \frac{1}{\tilde{m}_4^2} \left(2\ddot{\psi} - \frac{2c^2(\tilde{m}_2^2 - \tilde{m}_3^2)}{\tilde{m}_4^2} \Delta \psi + 2\tilde{m}_2^2 \psi + \frac{\tilde{m}_2^2 - \tilde{m}_3^2}{\tilde{m}_4^2} T_{00} - \frac{\ddot{T}_{00}}{a\Delta} \right). \quad (89)$$

Substituting these expressions into Eq.(86) together with $a = c^2$, one finds

$$\begin{aligned} & -6\ddot{\psi} + 8\left(\frac{1}{m_2^2} - \frac{1}{m_1^2}\right) \Delta \ddot{\psi} \\ & + c^2 \left[\left(8\frac{m_2^2}{m_4^2} - 2\right) \Delta \psi - 4\left(\frac{m_2^2 - m_3^2}{m_4^4}\right) \Delta^2 \psi - 6m_2^2 \psi \right] \\ & = 4\left(\frac{1}{m_4^2} - \frac{1}{m_1^2}\right) \frac{\ddot{T}_{00}}{c^2} - 2\left(\frac{m_2^2 - m_3^2}{m_4^4}\right) \Delta T_{00} - \frac{3}{\Delta} \frac{\ddot{T}_{00}}{c^2} + T_{ii} + 2\frac{m_2^2}{m_4^2} T_{00}. \end{aligned} \quad (90)$$

Once that time derivatives of ψ and its source T_{00} are neglected, the above equation could be solved to give

$$\psi = \frac{n_1 \Delta + n_0}{d_2 \Delta^2 + d_1 \Delta + d_0}, \quad (91)$$

where the n_i and d_i are polynomials in the masses.

The physics relevant to the vDVZ discontinuity is captured by expanding ψ in powers of $\frac{1}{\Delta}$ (that is, $\Delta \gg m_i^2$) as

$$\psi = \frac{T_{00}}{2\Delta} + \frac{c_1(m_i^2)}{\Delta^2} + \frac{c_2(m_i^2)}{\Delta^3} \quad (92)$$

in the $c = 1$ units. In the MLLV of (78) with $c_1(m_i^2) = c_2(m_i^2) = 0$, we obtain the same form as in (71)

$$\psi = \frac{1}{2\Delta} T_{00}, \quad (93)$$

which implies that there is no discontinuity at small distances. Also we find that E and B are finite in the massless limit. Therefore, there is no vDVZ discontinuity in the scalar sector.

However, for the Fierz-Pauli case, we find from (90) and (79) that ψ leads to

$$\psi = \frac{T_{ii} + 2T_{00} - \frac{3\ddot{T}_{00}}{c^2 \Delta}}{6c^2 \square}, \quad (94)$$

which takes a further form in the $c = 1$ units

$$\psi = \frac{T_{00}}{2\Delta} + \frac{T_{ii} - T_{00}}{6\square}. \quad (95)$$

This confirms the presence of the vDVZ discontinuity of the Fierz-Pauli case, as in Einstein gravity because the last term implies that ψ is a propagating degree of freedom. Consequently, we have shown that without the projectability condition, the Hořava scalar ψ is related to a scalar degree of freedom appeared in the massless limit of a massive graviton [7].

5 Discussions

We have studied graviton propagations of scalar, vector, and tensor modes in the dHL gravity (λR -model) without projectability condition. It is emphasized that the quadratic Lagrangian is invariant under diffeomorphism only for $\lambda = 1$ case. This contradicts to the fact that λ is irrelevant to a consistent Hamiltonian approach to the λR -model [17]. As far as scalar propagations are concerned, there is no essential difference between dHL gravity (λR -model) and general relativity. This implies that there are two degrees of freedom for a massless graviton without Hořava scalar, and five degrees of freedom including Hořava scalar appear for a massive graviton when introducing Lorentz-violating and Fierz-Pauli mass terms. Importantly, the strong coupling problem is not serious for vector and scalar modes when choosing Lorentz-violating mass term, as was claimed in general relativity [26]. It is shown that for $\lambda = 1$, the vDVZ discontinuity is absent in the massless limit of Lorentz-violating mass terms by considering external source terms. The dHL with $\lambda = 1$ recovers nicely the general relativity with (without) mass term in the linearized level. At this stage, we wish to distinguish the massless limit of massive dHL gravity from the $\lambda \rightarrow 1$ limit of dHL gravity. The former case provides one scalar (ψ) propagation as was shown in (95) when choosing the Fierz-Pauli mass term, while it provides no scalar propagation as was shown in (93) when choosing the Lorentz-violating mass term. The latter provides no scalar propagation, as was shown in (71).

On the other hand, the other case was the dHL gravity with the projectability condition, the $S_2^{\lambda R}$ is invariant under FDiff transformation for generic λ . In this case, the lapse perturbation $A(t)$ is not a Lagrange multiplier but a parameter [19]. The gauge transformations for modes is the same as in (28) and (29) except $B \rightarrow \tilde{B} = B + \dot{\xi}$ and thus, gauge-invariant scalars are ψ and $\Pi = 2B - \dot{E}$. However, this case gives rise to some difficulty in performing a consistent Hamiltonian analysis because the lapse perturbation A plays no role. For $1/3 < \lambda < 1$, the Hořava scalar ψ suffers from the ghost instability. Adding the Lorentz-violating mass term did not cure the ghost instability [19]. In order to avoid the ghost instability, one requires that the sound speed square c_ψ^2 be negative, leading to the gradient instability for $\lambda > 1$. To resolve this gradient instability, one has to impose the limit of $\lambda \rightarrow 1$, which leads to the strong coupling problem [10, 18]. However, it was suggested that there are many ways to tame the gradient instability of Hořava scalar [20].

Finally, we ask whether the projectability condition is really essential for being unable to rescue the dHL gravity from a doubtful modified gravity seriously. If one abandons an original constructing principle of the projectability inspired by condensed matter physics, one has likely found general relativity in the IR limit. However, we wish to point out

that although the λR -model has contributed to making a progress toward a consistent Hamiltonian approach to the dHL gravity, there remains a subtle issue on the equivalence between a gauge-fixed version of general relativity (λR -model) and general relativity (see footnote 1).

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